10. Další goniometrické rovnice
(s využitím goniom. vzorců)

Příklady

① Řešte v R

a) $2\cos^2 x = \sin x + 1$ $\qquad O = R,\ D = R$

$[\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x]$

$2(1-\sin^2 x) = \sin x + 1$

$2 - 2\sin^2 x = \sin x + 1$

$2\sin^2 x + \sin x - 1 = 0$

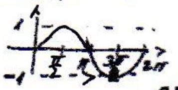
subst. $y = \sin x$

$2y^2 + y - 1 = 0$ $\quad (a=2, b=1, c=-1)$

$$y_{1/2} = \frac{-1 \pm \sqrt{1+4.2.1}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} \frac{2}{4} = \frac{1}{2} \\ -1 \end{cases}$$

$y_1 = \frac{1}{2}$ $\qquad$ zpětná subs. $\qquad$ $y_2 = -1$

$\sin x = \frac{1}{2}$ $\qquad\qquad$ $\sin x = -1$



$[x' = 30° = \frac{\pi}{6}]$

⊕ I., II. kv.

I. kv. $x_1 = \frac{\pi}{6} + 2k\pi$

II. kv. $x_2 = \pi - \frac{\pi}{6} + 2k\pi$

$x_2 = \frac{5\pi}{6} + 2k\pi$

$x_3 = \frac{3\pi}{2} + 2k\pi$

$$K = \bigcup_{k\in Z} \left\{\frac{\pi}{6} + 2k\pi, \frac{5\pi}{6} + 2k\pi, \frac{3\pi}{2} + 2k\pi\right\}$$

b) $\frac{1}{\cos^2 y} - \mathrm{tg}^2 y = 1$ $\qquad O = R$

$[\mathrm{tg}\, y = \frac{\sin y}{\cos y}]$

$\frac{1}{\cos^2 y} - \frac{\sin^2 y}{\cos^2 y} = 1$

$\frac{1 - \sin^2 y}{\cos^2 y} = 1$

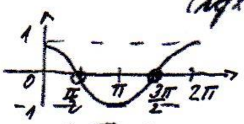
$[1 - \sin^2 y = \cos^2 y]$

$\frac{\cos^2 y}{\cos^2 y} = 1$

$1 = 1$

$K = D$

D: $\cos y \neq 0 \ \land \ D_{tg}: \cos y \neq 0$ $\quad (\mathrm{tg}\, x = \frac{\sin x}{\cos x})$



$x \neq \frac{\pi}{2} + k\pi$

$x \neq \frac{\pi}{2}(1+2k)$ $\quad$ liché nás. $\frac{\pi}{2}$

$$K = D = R - \bigcup_{k\in Z} \left\{\frac{\pi}{2}(2k+1)\right\}$$

c) $(\mathrm{tg}\, x)^{-1} - 2(\sin x)^{-1} = -\mathrm{tg}\, x$

$\frac{1}{\mathrm{tg}\, x} - \frac{2}{\sin x} = -\mathrm{tg}\, x$

$\mathrm{cotg}\, x - \frac{2}{\sin x} = -\mathrm{tg}\, x$

$\frac{\cos x}{\sin x} - \frac{2}{\sin x} = -\frac{\sin x}{\cos x} \quad / \cdot \sin x \cdot \cos x$

$\cos^2 x - 2\cos x = -\sin^2 x$

$\underbrace{\cos^2 x + \sin^2 x}_{1} - 2\cos x = 0$

$+2\cos x = +1$

$\cos x = \frac{1}{2}$

$[x' = 60° = \frac{\pi}{3}]$

⊕ I., IV. kv.

I. kv.

$x = x' + 2k\pi$

$x = \frac{\pi}{3} + 2k\pi \in D$

IV. kv.

$x = 2\pi - x' + 2k\pi$

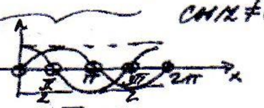
$x = 2\pi - \frac{\pi}{3} + 2k\pi$

$x = \frac{5\pi}{3} + 2k\pi \in D$

D: $\mathrm{tg}\, x \neq 0$ $\quad \sin x \neq 0$ $\quad D_{tg}: \mathrm{tg} = \frac{\sin x}{\cos x}$

$\frac{\sin x}{\cos x} \neq 0$ $\qquad \cos x \neq 0$

$\cos x \neq 0$



$x \neq 0 + k\frac{\pi}{2}$ nsk

$x \neq k\frac{\pi}{2}$

$$D = R - \bigcup_{k\in Z}\left\{k\frac{\pi}{2}\right\}$$

$$K = \bigcup_{k\in Z}\left\{\frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi\right\}$$

d) $\cos x - 1 = \sin^2 x$ $\qquad \mathcal{O} = R,\ \mathcal{D} = R$

2d

$\cos x - 1 = 1 - \cos^2 x$

$\cos^2 x + \cos x - 2 = 0$

subst. $y = \cos x$

$y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y_1 = -2 \qquad y_2 = 1$

zpět k subst.

$y_1 = -2$

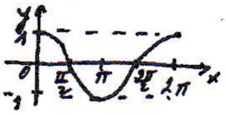
$\cos x = -2$

nelze

$\left[\begin{array}{l} y = \cos x \in \langle -1, 1\rangle \\ H_{\cos} = \langle -1, 1\rangle \\ -2 \notin H_{\cos} \end{array}\right]$

$y_2 = 1$

$\cos x = 1$



$x = 0 + 2k\pi$

$x = 2k\pi$

$K = \bigcup_{k \in Z} \{2k\pi\}$

② Řešte v R

a) $\sin x + \sqrt{3}\cos x = 1$

TYP: $a \sin x + b \cos x = c$

lze řešit různě, např. využ. $\sin^2 x + \cos^2 x = 1$

=> soustava 2 rovnic pro $\sin x, \cos x$ - lze subst.,

[nebo umocněním a převedením na 1 funkci

=> nek. úprava - zkouška nutná]

1. způ. subst. a alg. úpravy

$\sin x + \sqrt{3}\cos x = 1$

$\sin^2 x + \cos^2 x = 1$

subst. $u = \sin x,\ v = \cos x$

$u + \sqrt{3}v = 1 \Rightarrow u = 1 - \sqrt{3}v$ (*)

$u^2 + v^2 = 1$

$(1 - \sqrt{3}v)^2 + v^2 = 1$

$\cancel{1} - 2\sqrt{3}v + 3v^2 + v^2 = \cancel{1}$

$4v^2 - 2\sqrt{3}v = 0 \quad /:2$

$2v^2 - \sqrt{3}v = 0$

$v(2v - \sqrt{3}) = 0$

$v_1 = 0 \qquad 2v = \sqrt{3}$

$\qquad\qquad v = \frac{\sqrt{3}}{2}$

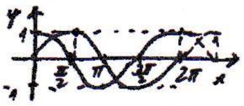
u(*) ! $u_1 = 1 - \sqrt{3}\cdot 0 \qquad u_2 = 1 - \sqrt{3}\cdot\frac{\sqrt{3}}{2}$

$u_1 = 1 \qquad u_2 = 1 - \frac{3}{2} = -\frac{1}{2}$

zpět k subst.

$u_1 = 1 \wedge v_1 = 0$

$\sin x = 1 \wedge \cos x = 0$



$x = \frac{\pi}{2} + 2k\pi$

$u_2 = -\frac{1}{2} \wedge v_2 = \frac{\sqrt{3}}{2}$

$\sin x = -\frac{1}{2} \qquad \cos x = \frac{\sqrt{3}}{2}$

⊖ III., IV. $\qquad$ ⊕ I., IV.

IV. kv.

$[x' = 30° = \frac{\pi}{6}]$

$x = 2\pi - \frac{\pi}{6} + 2k\pi$

$x = \frac{11\pi}{6} + 2k\pi$

$K = \bigcup_{k \in Z} \{\frac{\pi}{2} + 2k\pi, \frac{11\pi}{6} + 2k\pi\}$

2. způ. bez subst. alg. úpr.

$\sin x + \sqrt{3}\cos x = 1 \Rightarrow \sin x = 1 - \sqrt{3}\cos x \qquad \mathcal{O} = R,\ \mathcal{D} = R$

$\sin^2 x + \cos^2 x = 1$

$(1 - \sqrt{3}\cos x)^2 + \cos^2 x = 1$

$1 - 2\sqrt{3}\cos x + 3\cos^2 x + \cos^2 x = 1$

$4\cos^2 x - 2\sqrt{3}\cos x = 0$

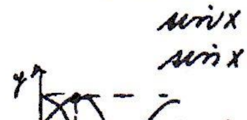
$2\cos^2 x - \sqrt{3}\cos x = 0$

$\cos x\,(2\cos x - \sqrt{3}) = 0$

! $\cos x = 0 \wedge \sin x = 1 - \sqrt{3}\cos x$

$\sin x = 1 - \sqrt{3}\cdot 0$

$\sin x = 1$



$x = \frac{\pi}{2} + 2k\pi$

$2\cos x - \sqrt{3} = 0 \wedge \sin x = 1 - \sqrt{3}\cos x$

$\cos x = \frac{\sqrt{3}}{2} \qquad \sin x = 1 - \sqrt{3}\cdot\frac{\sqrt{3}}{2}$

$\sin x = 1 - \frac{3}{2}$

⊕ I., IV. kv. $\qquad$ souč. $\qquad \sin x = -\frac{1}{2}$

⊖ III., IV. kv.

IV. kv. $\qquad [x' = \frac{\pi}{6} = 30°]$

$x = 2\pi - \frac{\pi}{6} + 2k\pi = \frac{11\pi}{6} + 2k\pi$

$K = \bigcup_{k \in Z} \{\frac{\pi}{2} + 2k\pi, \frac{11\pi}{6} + 2k\pi\}$

3. zp.) neekv. úpr. - umocn.

$\sin x + \sqrt{3}\cos x = 1$

$\sqrt{3}\cos x = 1 - \sin x \quad |^2$

$3\cos^2 x = 1 - 2\sin x + \sin^2 x$

$3(1 - \sin^2 x) = 1 - 2\sin x + \sin^2 x$

$3 - 3\sin^2 x = 1 - 2\sin x + \sin^2 x$

$4\sin^2 x - 2\sin x - 2 = 0$

$2\sin^2 x - \sin x - 1 = 0$

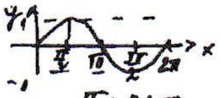
subst. $y = \sin x$

$2y^2 - y - 1 = 0$

$y_{1,2} = \frac{1 \pm \sqrt{1+8}}{4} = \begin{cases} 1 \\ -\frac{1}{2} \end{cases}$

zpět k subst.

$\sin x = 1$



$x_1 = \frac{\pi}{2} + 2k\pi$

$\sin x = -\frac{1}{2}$

$[x' = 30° = \frac{\pi}{6}]$

$\ominus$ III., IV. kv.

$x_2 = \pi + \frac{\pi}{6} + 2k\pi$

$x_2 = \frac{7\pi}{6} + 2k\pi$

$x_3 = 2\pi - \frac{\pi}{6} + 2k\pi$

$x_3 = \frac{11\pi}{6} + 2k\pi$

! Zkouška:

$L\left(\frac{\pi}{2} + 2k\pi\right) = \sin\left(\frac{\pi}{2} + 2k\pi\right) + \sqrt{3}\cos\left(\frac{\pi}{2} + 2k\pi\right) = \underbrace{\sin\frac{\pi}{2}}_{1} + \sqrt{3}\underbrace{\cos\frac{\pi}{2}}_{0} = 1 = P$

$L\left(\frac{7\pi}{6} + 2k\pi\right) = \sin\left(\frac{7\pi}{6} + 2k\pi\right) + \sqrt{3}\cos\left(\frac{7\pi}{6} + 2k\pi\right) = \sin\frac{7\pi}{6} + \sqrt{3}\cos\frac{7\pi}{6} = \sin 210° + \sqrt{3}\cos 210°$

$= -\sin 30° - \sqrt{3}\cos 30° = -\frac{1}{2} - \sqrt{3}\cdot\frac{\sqrt{3}}{2} = -\frac{1}{2} - \frac{3}{2} = -2 \neq P$! NENÍ KOŘEN

$L\left(\frac{11\pi}{6} + 2k\pi\right) = \sin\left(\frac{11\pi}{6} + 2k\pi\right) + \sqrt{3}\cos\left(\frac{11\pi}{6} + 2k\pi\right) = \sin\frac{11\pi}{6} + \sqrt{3}\cos\frac{11\pi}{6} =$

$= \underbrace{\sin 330°}_{\text{IV. kv.} -,\ \alpha' = 30°} + \sqrt{3}\underbrace{\cos 330°}_{\text{IV. kv.} +} = -\sin 30° + \sqrt{3}\cos 30° = -\frac{1}{2} + \sqrt{3}\cdot\frac{\sqrt{3}}{2} = -\frac{1}{2} + \frac{3}{2} = 1 = P$

$K = \bigcup_{k \in Z} \left\{\frac{\pi}{2} + 2k\pi, \frac{11\pi}{6} + 2k\pi\right\}$

b) $\sqrt{3}\sin x + \cos x = 2$

$\sqrt{3}\sin x + \cos x = 2$

$\sin^2 x + \cos^2 x = 1$

$O = R$
$D = R$

subst. $u = \sin x, v = \cos x$

$\sqrt{3}u + v = 2 \Rightarrow v = 2 - \sqrt{3}u \quad (*)$

$u^2 + v^2 = 1$ ✓

$u^2 + (2 - \sqrt{3}u)^2 = 1$

$u^2 + 4 - 4\sqrt{3}u + 3u^2 = 1$

$4u^2 - 4\sqrt{3}u + 3 = 0$

$u_{1,2} = \frac{4\sqrt{3} \pm \sqrt{16\cdot 3 - 4\cdot 4\cdot 3}}{8} = \frac{4\sqrt{3}}{8} = \frac{\sqrt{3}}{2}$

dop. (*) $v = 2 - \sqrt{3}\cdot\frac{\sqrt{3}}{2} = 2 - \frac{3}{2} = \frac{1}{2}$

zpět k subst.

$u = \frac{\sqrt{3}}{2}$ $\quad v = \frac{1}{2}$

$\sin x = \frac{\sqrt{3}}{2}$ $\quad \cos x = \frac{1}{2}$

$\oplus$ I., II. $\quad \oplus$ I., IV.

1. kv.

$x = x' + 2k\pi$ $\quad [x' = 60° = \frac{\pi}{3}]$

$x = \frac{\pi}{3} + 2k\pi$

$K = \bigcup_{k \in Z} \left\{\frac{\pi}{3} + 2k\pi\right\}$